Universal behaviour of neutron star multipole moments (I-Love-Q relations)

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NESTAR

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• Concurrence/Concordance in research on Universal properties

• Multipole moments in brief

• Neutron star structure and spacetime geometry
  – Numerical rotating neutron stars with realistic EOSs
  – Multipole moments of numerical spacetimes

• Universal relations between the moments

• Why I-Love-Q?

• Outlook
• **I-Love-Q in slow rotation** [K. Yagi, and N. Yunes, Science \textbf{341} 365 (2013)],
  A. Maselli et al., Phys.Rev. D \textbf{88} 023007 (2013),

  – and I-Q in rapid rotation

• **GW from NS mergers** [J. S. Read, et al., Phys.Rev. D \textbf{88} 044042 (2013)],


  – and Why I-Love-Q investigations

• **Universal relations in other settings:**
  - other...

• **in alternative theories of gravity** [K. Yagi, and N. Yunes, Science \textbf{341} 365 (2013)] (Chern-Simons),
Newtonian multipole moments:

\[ \Phi(r) = G \left( \frac{Q}{r} + \frac{Q_a x^a}{r^3} + \frac{Q_{ab} x^a x^b}{r^5} + \ldots \right) \]  \hspace{1cm} (1)

where, \( Q, Q_a, Q_{ab}, \) are some integrals on the source

\[ Q = \int \rho(r') d^3x', \quad Q_a = \int x'_a \rho(r') d^3x', \quad Q_{ab} = \int \frac{3}{2} (x'_a x'_b - \frac{1}{3} r'^2 \delta_{ab}) \rho(r') d^3x' \ldots \]  \hspace{1cm} (2)

The multipole moments are generally tensorial quantities.

Definition of the moments at infinity:

\[ x^a \rightarrow \tilde{x}^a = r^{-2} x^a; \quad \tilde{r}^2 = \tilde{x}^a \tilde{x}_a = r^{-2} \]

\[ \Phi(r) = \tilde{r} \left( Q + Q_a \tilde{x}^a + Q_{ab} \tilde{x}^a \tilde{x}^b + \ldots \right) \]  \hspace{1cm} (3)

If we define the potential at infinity \( \tilde{\Phi} = \tilde{r}^{-1} \Phi \) then the moments are

\[ P_{a_1 \ldots a_n} = \tilde{D}_{a_n} P_{a_1 \ldots a_{n-1}} = \tilde{D}_{a_1} \ldots \tilde{D}_{a_n} \tilde{\Phi} \]  \hspace{1cm} (4)
In General Relativity instead of a gravitational field $\Phi$, gravity is described by a metric $g_{ab}$.

**Relativistic multipole moments:**

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential", that is related to the metric, by a recursive relation,
- There are two sets of moments, the Mass moments and the Rotation moments,
- For the two sets of moments we have two generating potentials, $\Phi_M$, $\Phi_J$,

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.

The moments characterize the structure of the source and of the spacetime.
Relativistic non-rotating Stars

In spherical symmetry the metric can take the form $ds^2 = -e^{2\Phi}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}dr^2 + r^2d\Omega^2$.

Field equations: $G^{ab} = 8\pi GT^{ab}$,

Equation for the field $\Phi$: $\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 P}{r(r - 2m(r))}$,

Definition of the Mass: $\frac{dm}{dr} = 4\pi \rho r^2$,

Hydrostatic equilibrium: $\frac{dP}{dr} = -\left(\rho + P\right)\frac{d\Phi}{dr}$.

$$\frac{dP}{dr} = -\frac{\rho m(r)}{r^2} \left(1 + \frac{P}{\rho}\right) \left(1 + \frac{4\pi P r^3}{m(r)}\right) \left(1 - \frac{2m(r)}{r}\right)^{-1},$$

Equation of state for the fluid: $P = P(\rho)$.

The spacetime outside the star is the Schwarzschild spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2.$$
* SLB1,2 are observationally inferred EoSs

Relativistic rotating Stars

The line element for a stationary and axially symmetric spacetime (the spacetime admits a timelike, $\xi^a$, and a spacelike, $\eta^a$, killing field, i.e. it has rotational symmetry and symmetry in translations in time) is

$$ds^2 = -e^{2\nu}dt^2 + r^2 \sin^2 \theta B^2 e^{-2\nu} (d\phi - \omega dt)^2 + e^{2(\zeta-\nu)} (dr^2 + r^2 d\theta^2),$$

Field equations in the frame of the ZAMOs:

$$D \cdot (BD\nu) = \frac{1}{2} r^2 \sin^2 \theta B^3 e^{-4\nu} D\omega \cdot D\omega + 4\pi B e^{2\zeta-2\nu} \left[ \frac{(\epsilon + p)(1 + u^2)}{1 - u^2} + 2p \right],$$

$$D \cdot (r^2 \sin^2 \theta B^3 e^{-4\nu} D\omega) = -16\pi r \sin \theta B^2 e^{2\zeta-4\nu} \frac{(\epsilon + p)u}{1 - u^2},$$

$$D \cdot (r \sin \theta DB) = 16\pi r \sin \theta B e^{2\zeta-2\nu} p,$$

Komatsu, Eriguchi, and Hechisu\(^2\) proposed a scheme for integrating the field equations using Green’s functions. This scheme is implemented in the RNS numerical code to calculate rotating neutron stars\(^3\).


One can use RNS to calculate models of rotating neutron stars for a given equation of state. For example we show here some models for the APR EOS:

The models with the fastest rotation have a spin parameter, \( j = J/M^2 \), around 0.7 and a ratio of the polar radius over the equatorial radius, \( r_p/r_e \), around 0.56.

The code, except from the various physical characteristics of the neutron stars, provides the metric functions in a grid on the coordinates \( x \) and \( \mu \) in the whole space (for values from 0 to 1 for both variables), where \( \mu = \cos \theta \), \( r = \frac{xr_e}{1-x} \) and \( r_e \) is a length scale.
From the field equations we have the asymptotic expansion:

\[
\nu = \left\{ -\frac{M}{r} + \frac{1}{3} \frac{\tilde{B}_0}{r^3} + \frac{J^2}{r^4} + \left[ \frac{-\tilde{B}_0}{5} + \frac{\tilde{B}_2}{15} - \frac{12J^2}{5} \right] \frac{M}{r^5} + \ldots \right\} \\
+ \left\{ \frac{Q_2}{r^3} - \frac{2J^2}{r^4} + [\ldots] \frac{1}{r^5} + \ldots \right\} P_2(\mu) \\
+ \left\{ \frac{Q_4}{r^5} + \ldots \right\} P_4(\mu) + \ldots
\]

\[
\omega = \left\{ \frac{2J}{r^3} - \frac{6JM}{r^4} + \frac{6}{5} \left[ 8 - 3 \frac{\tilde{B}_0}{M^2} \right] \frac{JM^2}{r^5} + (\ldots) \frac{J}{r^6} + \ldots \right\} P_{1,\mu}(\mu) \\
+ \left\{ \frac{W_2}{r^5} + (\ldots) \frac{1}{r^6} - \ldots \right\} P_{3,\mu}(\mu) + \ldots
\]

\[
B = \left( \frac{\pi}{2} \right)^{1/2} \left( 1 + \frac{\tilde{B}_0}{r^2} \right) T_0^{1/2}(\mu) + \left( \frac{\pi}{2} \right)^{1/2} \frac{\tilde{B}_2}{r^4} T_2^{1/2}(\mu) + \ldots
\]

where \( P_l \) are the Legendre polynomials, \( \mu = \cos \theta \), and \( T_l^{1/2} \) are the Gegenbauer polynomials.
Multipole moments of numerical spacetimes

The RNS code also calculates the first non-zero multipole moments, i.e., \( M, S_1, M_2, S_3 \) and \( M_4 \). \(^4\)

\[
\begin{align*}
M_0 &= M, \\
S_1 &= j M^2, \\
M_2 &= -\frac{1 + 4 b_0 + 3 q_2}{3} M^3, \\
S_3 &= -\frac{3(2 j + 8 j b_0 - 5 w_2)}{10} M^4, \\
M_4 &= \frac{19 - 18 j^2 + 160 b_0 + 120 q_2 + 336 b_0^2 + 360 b_0 q_2 - 105 q_4 - 192 b_2}{105} M^5
\end{align*}
\]

where \( j \equiv J/M^2 \), and the various parameters are given by the integrals,

\[
\begin{align*}
Q_{2l} &= M^{2l+1} q_{2l} = -\frac{r_e^{2l+1}}{2} \int_0^1 \frac{ds' s'^{2l}}{(1 - s')^{2l+2}} \int_0^1 d\mu' P_{2l}(\mu') S_\rho(s', \mu'), \\
W_{2l-2} &= M^{2l} w_{2l-2} = -\frac{r_e^{2l}}{4l} \int_0^1 \frac{ds' s'^{2l}}{(1 - s')^{2l+2}} \int_0^1 d\mu' \sin \theta' P_{2l-1}(\mu') S_\omega(s', \mu'), \\
\bar{B}_{2l} &= M^{2l+2} b_{2l} = -\frac{16 \sqrt{2} \pi r_e^{2l+4}}{2l + 1} \int_0^{1/2} \frac{ds' s'^{2l+3}}{(1 - s')^{2l+5}} \int_0^1 d\mu' \sin \theta' p(s', \mu') B e^{2(\zeta - \nu)} T_{2l}^{1/2}(\mu'),
\end{align*}
\]

where in the second integral \( l \geq 1 \), in contrast to the other two integrals where \( l \geq 0 \).

Neutron star multipole moments properties

Black Hole-like behaviour of the moments$^5$:

<table>
<thead>
<tr>
<th>Kerr moments</th>
<th>Neutron star moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0 = M$,</td>
<td>$M_0 = M$,</td>
</tr>
<tr>
<td>$S_1 = J = jM^2$,</td>
<td>$S_1 = jM^2$,</td>
</tr>
<tr>
<td>$M_2 = -j^2M^3$,</td>
<td>$M_2 = -a(EoS,M)j^2M^3$,</td>
</tr>
<tr>
<td>$S_3 = -j^3M^4$,</td>
<td>$S_3 = -\beta(EoS,M)j^3M^4$,</td>
</tr>
<tr>
<td>$M_4 = j^4M^5$,</td>
<td>$M_4 = \gamma(EoS,M)j^4M^5$,</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$M_{2n} = (-1)^nj^{2n}M^{2n+1}$,</td>
<td>$M_{2n} = \text{?}$,</td>
</tr>
<tr>
<td>$S_{2n+1} = (-1)^nj^{2n+1}M^{2n+2}$</td>
<td>$S_{2n+1} = \text{?}$</td>
</tr>
</tbody>
</table>

Frequencies for the spacetime around neutron stars: Effect of the quadrupole*

Orbital frequency, periastron precession frequency and nodal precession frequency:
Orbital, $M\nu_\phi$, and precession, $M\nu_a$, “scaled frequencies” for neutron star models constructed with the APR EOS for approximately the same $j = 0.39$ and different higher moments.

Neutron star multipole moments properties

EoS independent behaviour of the moments$^6$:

$$\bar{M}_{2n} = \left| \frac{M_{2n}}{(j^{2n}M^{2n+1})} \right|, \quad \bar{S}_{2n+1} = \left| \frac{S_{2n+1}}{(j^{2n+1}M^{2n+2})} \right|$$

Neutron star multipole moments properties

Different polytropes\(^7\)

I-Q relation for rapidly rotating stars\(^8\)

\[ \chi \equiv j \equiv J/M^2 \]

\(^8\)G.P. and T. A. Apostolatos, Phys.Rev.Lett. 112 121101 (2014)].
Applications:

- Identification of the EoS
  - Overcoming the degeneracies by reducing the parameters (GWs, E/M signals)
  - ...

- Testing GR - distinguishing between alternative theories of gravity
  - A word of caution: Multipole Moments need to be properly defined in alternative theories of gravity
  - Compare carefully chosen corresponding parameters between GR and other theories
An Example: A “measurement” of the first 3 moments ($M, J, Q$), or a combination of other parameters, could select an EoS$^9$

Why do these universal relations exist?

- Where does the EoS independence come from?
  
  - The realistic EoSs are too similar
  
  - The structure of the star (matter distribution)
  
  - Property of gravity
  
  - …
  
  - Accident/Chance
**Newtonian insight**\(^{10}\)

In the Newtonian limit (weak field limit), mass and angular momentum moments can be written as:

\[
M_l = 2\pi \int_0^\pi \int_0^{R(\theta)} \rho(r, \theta) P_l(\cos \theta) \sin \theta d\theta dr^{l+2},
\]

\[
S_l = \frac{4\pi}{l+1} \Omega \int_0^\pi \int_0^{R(\theta)} \rho(r, \theta) \frac{dP_l(\cos \theta)}{d \cos \theta} \sin^3 \theta d\theta dr^{l+3}.
\]

Assuming constant ellipticity \(e\):

\[
M_l = 2\pi I_{l,3} R_l,
\]

\[
S_l = \frac{4\pi l}{l+1} \Omega (I_{l-1,5} - I_{l+1,3}) R_{l+1},
\]

where,

\[
I_{l,3} = (-1)^{l/2} \frac{2}{l+1} \sqrt{1 - e^2 e^l},
\]

\[
I_{l-1,5} - I_{l+1,3} = (-1)^{(l-1)/2} \frac{2(2l+1)}{l(l+2)} \sqrt{1 - e^2 e^{l-1}},
\]

\[
R_l = \int_0^{a_1} \rho(\bar{r}) \bar{r}^{l+2} d\bar{r},
\]

and \(\bar{r}\) is the coordinate radius in the elliptic coordinate system. The mean radius of the surface of the star is \(R = a_1 (1 - e^2)^{1/6}\), where \(a_1\) is the semi-major axis and corresponds to the \(\bar{r}\) at the surface.

For polytropes:

\[
\bar{M}_{2l+2} = \bar{M}_2 \bar{S}_{2l+1}, \quad \bar{M}_{2l+2} = \bar{A}_{n,l} (\bar{S}_{2l+1})^{1+1/l} \Rightarrow
\]

\[
\bar{M}_{2l+2} + i \bar{S}_{2l+1} = (\bar{A}_{n,l})^{-l} \bar{M}_2^l (\bar{M}_2 + i \bar{S}_1).
\]

Newtonian insight

Assumptions: Newtonian polytropes ($P = K \rho^{1+1/n}$), Constant ellipticity for the isodensity surfaces.

Result: Approximately universal relations as long as the variation in $\bar{A}_{n,l}$ is small.
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But the ellipticity is not constant for $n \neq 0$:

So, is this a good approximation?

Additionally, realistic EoSs are not simple polytropes, and can be better described as piece-wise polytropes.

Beyond simple Newtonian insight (the EoS)

Piecewise polytropic EoS in the Hartle-Thorne approximation to quadratic order in spin:

\[ \Gamma_i = 1 + 1/n_i \]
Beyond simple Newtonian insight (the EoS)

Piecewise polytropic EoS in the Hartle-Thorne approximation to quadratic order in spin:

\[ \Gamma_i = 1 + 1/n_i \]

Although the variation of the EoS parameters is not small, the variation in the \( \bar{I} - \bar{Q} \) relation is small. The variation in the EoS is not too small even if one restricts oneself in the region most important for \( \bar{I} \) and \( \bar{Q} \) (i.e., \( 14 < \log_{10} \rho < 15 \)). In any case, \( \bar{n} \) is always, \( 0.5 < \bar{n} < 1 \).
Beyond simple Newtonian insight (the ellipticity)

Variation of the ellipticity of isodensity contours:
– How much does the ellipticity vary?
– What is the effect?
Beyond simple Newtonian insight (the ellipticity)

Variation of the ellipticity of isodensity contours:
– How much does the ellipticity vary?
– What is the effect?

\[ \lambda = \frac{p_c}{\varepsilon_c} \] is a measure of how relativistic the star is. Top \( n = 0.5 \). Bottom \( n = 1 \).
Beyond simple Newtonian insight (the ellipticity)

Change in the coefficients of the Newtonian universal relations for realistic ellipticity profiles:

If one assumes that the ellipticity is a radial function of the form, \( e(r) = e_0 f(r) \), one can derive relations between the moments as before,

\[
\bar{M}_{2l+2} = \bar{B}_{n,l}^{(f)} \bar{M}_2 \bar{S}_{2l+1}, \quad \bar{M}_{2l+2} = \bar{A}_{n,l}^{(f)} (\bar{S}_{2l+1})^{1+1/l}
\]
Conclusion

- Properties of matter, in densities above nuclear, give EoSs that have polytropic index $n$ in the interval $[0, 1]$.

- Such EoSs produce matter distributions that are not centrally condensed and even GR corrections don’t change that much.

- A density profile that is not centrally condensed produces small variation of the ellipticity throughout the star.

- The combination of the above, result to relations between the multipole moments that are not sensitive to the realistic EoSs differences.

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- The combination of the above, result to relations between the multipole moments that are not sensitive to the realistic EoSs differences.

Corroborating result by G. Martinon, et al., arXiv:1406.7661 [gr-qc]

• Extending our understanding to the other types of universality and relating the various universalities in a bigger picture

• On the EoS front
  – Selection of potentially useful astrophysical observables
  – Model the implementation of universal relations

• On the tests of gravity front
  – Exploration of corresponding universal relations in alternative theories of gravity (already work has started on this)
  – Selection of potentially useful observables
Thank You.